Journal of Sound and Vibration (1998) **218**(3), 389–403
Article No. sv981864

THE INSTANTANEOUS LYAPUNOV EXPONENT AND ITS APPLICATION TO CHAOTIC DYNAMICAL SYSTEMS

K. SHIN AND J. K. HAMMOND
ISVR, University of Southampton, Highfield, Southampton SO17 1BJ, England

(*Received* 23 *October* 1996, *and in final form* 2 *June* 1998)

Any system containing at least one positive Lyapunov exponent is defined to be chaotic and the system dynamics become unpredictable. For a mechanical system, the sum of Lyapunov exponents is negative and related to the damping, and so can be utilised to monitor any changes of the damping mechanism. However, in order to track any changes the data segment used to compute the Lyapunov exponents must be short. This leads to problems since Lyapunov exponents are calculated from a long term averaged divergence rate. To overcome this problem, it is described how the sum of Instantaneous Lyapunov exponents is related to the generalised divergence of the flow and the damping of a mechanical system. Computer simulations using differential equations are implemented to demonstrate the significance of the sum of Instantaneous Lyapunov exponents. In practice, it may be difficult to obtain accurate Instantaneous Lyapunov exponents from a time series (from experimental data), and so short term averaged Lyapunov exponents are introduced to overcome this. The sum of short term averaged Lyapunov exponents is applied to experimental data to detect evolutionary changes in damping from high to low.

1. INTRODUCTION

In a dynamical system, the spectrum of Lyapunov exponents plays a very important role in the diagnosis of whether the system is chaotic or not. The Lyapunov exponents are a measure of sensitive dependence upon initial conditions and represent the average rate of divergence or convergence of nearby trajectories in phase space. A positive Lyapunov exponent means that the nearby trajectories in phase space will soon diverge and the evolution is sensitive to initial conditions and the system becomes unpredictable. Any system containing at least one positive Lyapunov exponent is defined to be chaotic [1]. One of the important properties is that the sum of Lyapunov exponents is related to the generalised divergence of the flow in phase space of the system, and related to the energy dissipation mechanism of a system, i.e., the energy dissipation means phase volume contraction. In a mechanical system damping is related to the energy dissipation. Thus the sum of Lyapunov exponents must be related to the damping of a mechanical system, and can be utilised to monitor any changes of damping of the

0022–460X/98/480389 + 15 $30.00/0

system. However, in order to track the changes of damping of a system the data segment used to calculate the Lyapunov exponents must be short. This leads to problems since Lyapunov exponents are calculated from a long term averaged divergence rate as will be discussed later.

To overcome the above problem, Instantaneous Lyapunov exponents (ILEs) are introduced which are the derivative of the logarithm of divergence rate. It is described how the sum of ILEs is related to the generalised divergence of the flow and to the damping of a mechanical system. Computer simulation results from differential equations are presented, and experimental results using a double-well potential vibrator are also presented. The algorithm for computing the Lyapunov exponents (and the ILEs) from differential equations is based on the use of a phase space plus tangent space approach suggested by Wolf *et al*. [2], and on the algorithm developed by Sano *et al*. [3] in the case of experimental data (time series). In practice, it is very difficult to obtain accurate ILEs from a time series due to computational errors, and so short term averaged Lyapunov exponents (SLEs) are introduced. A similar concept to the SLEs, the 'Local Lyapunov exponents' was introduced by Abarbanel *et al*. [4]. However, the approach here is different, i.e., 'Local Lyapunov exponents' are defined in a discrete manner, and so, strictly speaking, are only relevant for discrete-time dynamical systems. For an example of the Local Lyapunov exponents for a discrete-time dynamical system, Lyapunov exponents calculated from the nth interations of the system refer to the local Lyapunov exponent of order n. For continuous-time dynamical systems, they defined the Local Lyapunov exponents in the same way using a sampled version of the system. However, they did not discuss any effects of time-discretisation of continuous systems. On the other hand, the SLE is the time averaged version of the ILE, so it is defined in a continuous manner for continuous-time dynamical systems and is the quantity which varies locally in time. Thus the SLE is very different from the local Lyapunov exponent in this sense.

In this paper, the generalised divergence of the flow in phase space will be discussed and the ILEs and SLEs derived in section 2. The relationship between the damping property of a mechanical system and the generalised divergence of the flow is discussed in section 3. Computer simulation results using simple non-linear differential equations (Duffing equation and Van der Pol equation) are presented in section 4. Experimental results are given in section 5.

2. VOLUME IN A PHASE SPACE AND THE INSTANTANEOUS LAYAPUNOV EXPONENTS

A continuous-time n-dimensional autonomous dynamical system is modelled by ordinary differential equations of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(x_1, x_2, \ldots, x_n) = \left[\frac{\mathrm{d}x_1}{\mathrm{d}t}\,\frac{\mathrm{d}x_2}{\mathrm{d}t}\,\cdots\,\frac{\mathrm{d}x_n}{\mathrm{d}t}\right]^{\mathrm{T}}, \quad \mathbf{x}(t_0) = \mathbf{x}_0,\, \mathbf{x}(t) \in \Re^n \tag{1}$$

where $\mathbf{x} = [x_1\, x_2\, \cdots\, x_n]^{\mathrm{T}}$ is a vector in n-dimensional phase space and x_i are phase coordinates. Equation (1) determines the set of solution curves (trajectories or flow) in phase space. The vector function $\mathbf{f}$ is the generalised velocity vector field associated with flow. Suppose that the long-term evolution of an infinitesimal n-dimensional sphere of initial conditions is monitored, the sphere will become an n-dimensional ellipsoid due to the locally deforming nature of the flow (the flow is a bundle of trajectories). The ith Lyapunov exponent in the n-dimensional phase space is defined in terms of the length of the ith principal axis $P_i(t)$ of the ellipsoid [5]

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{P_i(t)}{P_i(0)} \tag{2}$$

where the λ_i are ordered from the largest to the smallest. The Lyapunov exponents defined in the form of (2) is not very helpful in computation, because it is impossible to average infinitely. Thus the Lyapunov exponent may be described by an estimate which becomes a function of time or a function of the number of iterations. Then equation (2) becomes

$$\lambda_i(t) = \frac{1}{t} \ln \frac{P_i(t)}{P_i(0)}\,. \tag{3}$$

Similarly, if the system is described by difference equations or a map, then the Lyapunov exponents are defined in a similar manner to the continuous-time system as

$$\hat{\lambda}_i(k) = \frac{1}{k} \ln \frac{P_i(k)}{P_i(0)} \tag{4}$$

where $P_i(k)$ is the length of the ith principal axis, and k is the number of iterations of the system. In this paper, assuming the estimates of the Lyapunov exponents are obtained from long-term averaged values, the Lyapunov exponents (λ_i) usually mean the *estimates* of the Lyapunov exponents, so generally the notation in (3) and (4) will not be used explicitly, unless otherwise stated.

If one of the λ_i is positive, nearby trajectories diverge, and the evolution is sensitive to initial conditions and is therefore chaotic. However, the divergence of chaotic trajectories can only be locally exponential, because if the system is bounded $P_i(t)$ cannot go to infinity. Thus to obtain the Lyapunov exponents, one must average the local exponential growth over a long time (infinite in theory). Lyapunov exponents have been defined in terms of the principal axes of an n-dimensional ellipsoid in an n-dimensional phase space. Similarly, the behaviour of the volume of the ellipsoid is related to the sum of Lyapunov exponents. The relative rate of change of a n-dimensional volume V in n-dimensional phase space under the action of flow is given by the 'Lie derivative' (generalised divergence of the flow) [6, 7].

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}\,. \tag{5}$$

This relative rate of volume change can also be expressed by the Lyapunov exponents (λ_i) [8]

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_{i=1}^{n} \lambda_i. \tag{6}$$

Thus, the sum of Lyapunov exponents is equal to the generalised divergence of the flow. However, this is not a rigorous relationship since the Lyapunov exponents are obtained from the long-term averaged divergence rate [see equation (2)], whereas the generalised divergence of the flow may continuously change with the action of the flow (e.g., the Van der Pol equation, which will be shown later). This may be much more rigorously described by introducing the ILEs. The divergence rate $P_i(t)/P_i(0)$ in equation (3) is continuous in time. Since the trajectories obtained from equation (1) are smooth and bounded, the divergence rate is also smooth and bounded. Thus the divergence rate is differentiable. The ILE for a continuous-time dynamical system is defined as the derivative of the logarithm of the divergence rate

$$\alpha_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln \frac{P_i(t)}{P_i(0)}\right]. \tag{7}$$

Also, the ILE for a discrete-time dynamical system (difference equation) is more easily defined such that

$$\alpha_i(k) = \ln \frac{P_i(k)}{P_i(k-1)} \tag{8}$$

where $\alpha_i(k)$ is the ith ILE at the kth iteration of the system, and represents the divergence rates at each iteration. In this paper, continuous-time dynamical systems are considered. Unlike the Lyapunov exponent, the ILE represents the divergence (or convergence) rate at a given time and is a time varying quantity. So the ILE shows how the divergence (or convergence) rate of nearby trajectories changes with time. The Lyapunov exponent becomes the time average of the ILE

$$\hat{\lambda}_i(t) = \frac{1}{t}\int_0^t \alpha_i(t_1)\,\mathrm{d}t_1 \tag{9}$$

or

$$\hat{\lambda}_i(t) = A[\alpha_i(t)] \tag{10}$$

where $A[\,]$ denotes the time average. Thus, the length of the ith principal axis at time t can be described by the ILE,

$$P_i(t) = P_i(0)\exp\left[\int_0^t \alpha_i(t_1)\,\mathrm{d}t_1\right] \tag{11}$$

and the n-dimensional phase volume at time t becomes

$$V(t) = V(t_0)\exp\left[\sum_{i=1}^{n}\int_0^t \alpha_i(t_1)\,\mathrm{d}t_1\right] = V(t_0)\exp\left(A\left[\sum_{i=1}^{n}\alpha_i\,(t_1)\right]t\right) \tag{12}$$

let $\sum_{i=1}^{n}\alpha_i\,(t) = \sigma(t)$ for convenience, i.e., $\sigma(t)$ is the sum of ILEs. Then, equation (12) becomes

$$V(t) = V(t_0)\exp\left[\int_0^t \sigma(t_1)\,\mathrm{d}t_1\right] = V(t_0)\exp(A[\sigma(t_1)]t)$$

$$\therefore \frac{1}{V(t)}\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \sigma(t) \tag{13}$$

where $\sigma(t) = \mathrm{d}/\mathrm{d}t(A[\sigma(t_1)]\cdot t + A[\sigma(t_1)])$, and $A[\sigma(t)] = \sum_{i=1}^{n}\hat{\lambda}_i(t)$. Thus, from equation (13), the sum of ILEs is equal to the generalised divergence of the flow. And the sum of ILEs represents the instantaneous behaviour of the generalised divergence of the flow. The SLE is defined as the short-term average of the ILE, i.e., ILEs are averaged over a defined window length to calculate the SLEs.

$$S_i(t) = A[\alpha_i(t)]_{t_1}^{t_1+t_w} \tag{14}$$

where t_w is the window length. The relationship between the ILE and SLE is graphically illustrated in Figure 1.

The SLE is introduced especially for the case of experimental data (time series), because the ILEs obtained from the algorithm (linear approximation of the flow) based on Sano *et al*. [3] are subject to ‘numerical’ noise. By assuming that the numerical noise is uncorrelated, the noise can be reduced by taking an average (SLEs). The size of window length may depend on the application. For the particular application presented in this paper, the window length is determined by trial and error, i.e., striking a balance between the detectability of changes of damping and numerical errors as will be shown later.

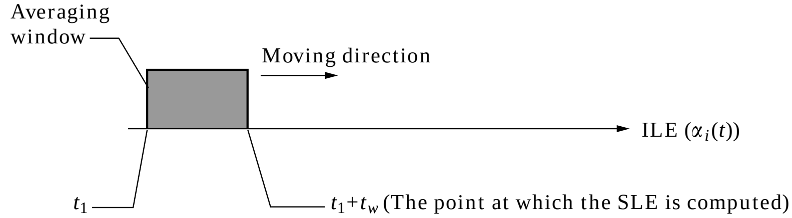

Figure 1. Calculation of SLE from ILE.

3. GENERALISED DIVERGENCE OF THE FLOW AND THE DAMPING PROPERTY OF A MECHANICAL SYSTEM

The equation of motion of a linear multi-degree of freedom system can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{15}$$

where $\mathbf{M}$ is the diagonal mass matrix, $\mathbf{C}$ and $\mathbf{K}$ are the damping matrix and stiffness matrix, and $\mathbf{x}$ and $\mathbf{f}(t)$ are n-dimensional vectors. If the system is described in $2n$-dimensional non-autonomous phase space, by letting $\mathbf{p} = \mathbf{x}$ and $\mathbf{q} = \dot{\mathbf{x}}$, the equations of motion may be written as

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{q} = \mathbf{f}_1(p_1, \ldots, p_n, \ldots, q_n, t)$$

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{M}^{-1}\mathbf{f}(t) - \mathbf{M}^{-1}\mathbf{C}\mathbf{q} - \mathbf{M}^{-1}\mathbf{K}\mathbf{p} = \mathbf{f}_2(p_1, \ldots, p_n, q_1, \ldots, q_n, t) \tag{16}$$

where $\mathbf{f}_1 = (f_1, \ldots, f_n)$ and $\mathbf{f}_2 = (f_{n+1}, \ldots, f_{2n})$. Since the mass matrix is diagonal, the generalised divergence of the flow becomes

$$\sum_{k=1}^{n} \frac{\partial f_k}{\partial p_k} + \sum_{k=n+1}^{2n} \frac{\partial f_k}{\partial q_k} = -\mathrm{trace}(\mathbf{M}^{-1}\mathbf{C}). \tag{17}$$

This can be interpreted as the total damping of the system. As a result, in a linear system, the generalised divergence of the flow is independent of the external force (assuming that the external forces are harmonic) and the restoring force, and thus represents the viscous damping property of the system. If a single degree of freedom system is considered, the generalised divergence of the flow becomes $-c/m$, where c is damping coefficient and m is mass.

For some non-linear systems, the generalised divergence of the flow may change continuously. Note that equation (17) does not hold for non-linear systems since the damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ may be functions of $\mathbf{p}$ and $\mathbf{q}$, and the partial derivatives in (17) depend on the non-linearities of the system. The equation of motion of a class of non-linear multi-degree of freedom system may be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}(\mathbf{x}, \dot{\mathbf{x}})\dot{\mathbf{x}} + \mathbf{K}(\mathbf{x}, \dot{\mathbf{x}})\mathbf{x} = \mathbf{f}(t). \tag{18}$$

By letting $\mathbf{p} = \mathbf{x}$ and $\mathbf{q} = \dot{\mathbf{x}}$, the equations of motion may be written as

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{q} = \mathbf{f}_1(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$$

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{M}^{-1}\mathbf{f}(t) - \mathbf{M}^{-1}\mathbf{C}(\mathbf{p}, \mathbf{q})\mathbf{q} - \mathbf{M}^{-1}\mathbf{K}(\mathbf{p}, \mathbf{q})\mathbf{p} = \mathbf{f}_2(p_1, \ldots, p_n, q_1, \ldots, q_n, t). \tag{19}$$

The generalised divergence of the flow becomes

$$\sum_{k=1}^{n} \frac{\partial f_k}{\partial p_k} + \sum_{k=n+1}^{2n} \frac{\partial f_k}{\partial q_k} = -\text{sum}\left\{\frac{\partial}{\partial \mathbf{q}}(\mathbf{M}^{-1}\mathbf{C}(\mathbf{p},\mathbf{q})\mathbf{q})\right\} - \text{sum}\left\{\frac{\partial}{\partial \mathbf{q}}(\mathbf{M}^{-1}\mathbf{K}(\mathbf{p},\mathbf{q})\mathbf{p})\right\} \tag{20}$$

where sum {} denotes the sum of all elements of the vector. From (20), it can be shown that the generalised divergence may not only depend on **C** but also depends on **K** when different modes interact with each other. However, if **K** is a pure stiffness matrix (i.e., function of **p** only **K**(**p**)), the second term of the right-hand side of (20) disappears. In general **K**(**p**, **q**) is not a pure stiffness term but includes damping as well, so one may still say that the generalised divergence represents the total damping property of the system.

4. NUMERICAL SIMULATIONS USING DIFFERENTIAL EQUATIONS

The algorithms in references [2, 3] are modified to compute the ILE, and the details of the computational aspects of ILE can be found in [9, 10]. The simulation is focused on the possibility of detecting changes of damping of a non-linear (chaotic) mechanical system. Two differential equations are investigated—Duffing and Van der Pol equations. Simulations are conducted in which the damping parameter is changed at a certain time. Comparisons are made of changes in the sum of ILEs and the sum of Lyapunov exponents. These results are also compared to the sum of SLEs.

EXAMPLE 1

The Duffing equation ($\ddot{x}_1 + c\dot{x}_1 - kx_1(1 - x_1^2) = A\cos\omega t$) is considered which can be expressed in the form (1).

$$\frac{dx_1}{dt} = x_2 = f_1(x_1, x_2, x_3)$$

$$\frac{dx_2}{dt} = -cx_2 + kx_1(1 - x_1^2) + A\cos(x_3) = f_2(x_1, x_2, x_3)$$

$$\frac{dx_3}{dt} = \omega = f_3(x_1, x_2, x_3) \tag{21}$$

and, the generalised divergence of the flow is equal to the negative of the damping coefficient which is constant,

$$\sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} = \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right] = -c. \tag{22}$$

This implies that the sum of ILEs is equal to the negative of the damping coefficient and does not vary with time. The forcing parameter and stiffness parameter are fixed and two different damping parameters are chosen in the simulation (Figure 2). The phase portraits for these systems are shown in

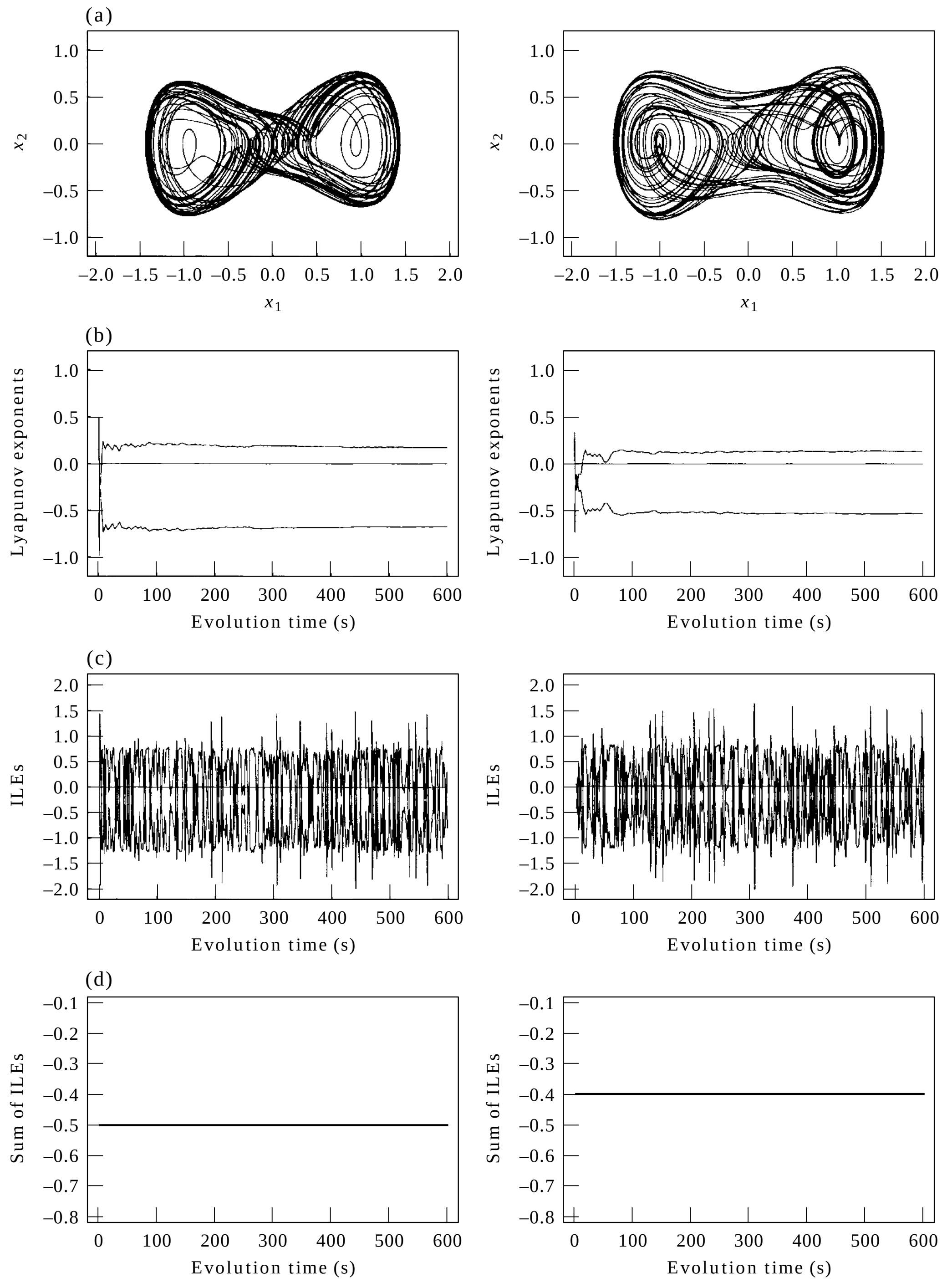

Figure 2. Results of the simulation for the Duffing equation. (left: $A = 0{\cdot}4$, $k = 1$, $\omega = 1$, $c = 0{\cdot}5$, and right $A = 0{\cdot}4$, $k = 1$, $\omega = 1$, $c = 0{\cdot}4$). (a) Phase portraits. (b) Lyapunov exponents for each case: both have a positive Lyapunov exponent. (c) ILEs for each case: ILEs are fluctuating caused by the continuously varying local divergence rate of the nearby trajectories. (d) Sum of ILEs for each case: they are equal to $-0{\cdot}5$ (left-hand side of figure) and $-0{\cdot}4$ (right-hand side of the figure), and represent damping properties, and these show that the rate volume contraction is always the same.

Figure 2(a), and the positive Lyapunov exponent shows that both systems are chaotic [Figure 2(b)]. The corresponding ILEs are also shown in Figure 2(c). The ILEs looks apparently random, however the sum of ILEs is always the same and equal to the generalised divergence of the flow, and equal to the negative of the damping coefficient as shown in Figure 2(d). Clearly the sum of Lyapunov exponents is the same as the sum of ILEs, so it is not shown in the figures. As a result, it can be said that the sum of ILEs represents the damping property. Now, we consider the system in which the dampng parameter is 0·5 and is changed to 0·4 at a certain time (at 600 s for this example). The results of this system are shown in Figure 3. In Figure 3(a), it is shown that the sum of Lyapunov exponents varies very slowy when the damping parameter is changed, and so it is very difficult to see whether there is any change at this time. On the other hand, in Figure 3(b), the sum of ILEs is clearly distinguishable at the right point when the damping parameter is changed by changing the value from $-0{\cdot}5$ to $-0{\cdot}4$. This shows how the sum of ILEs can effectively be used for detecting changes of damping of a non-linear (chaotic) mechanical system. The sum of SLEs is presented in Figure 3(c), demonstrates the ability to detect changes of damping. The SLE is obtained by averaging 10 previous forcing periods of ILEs.

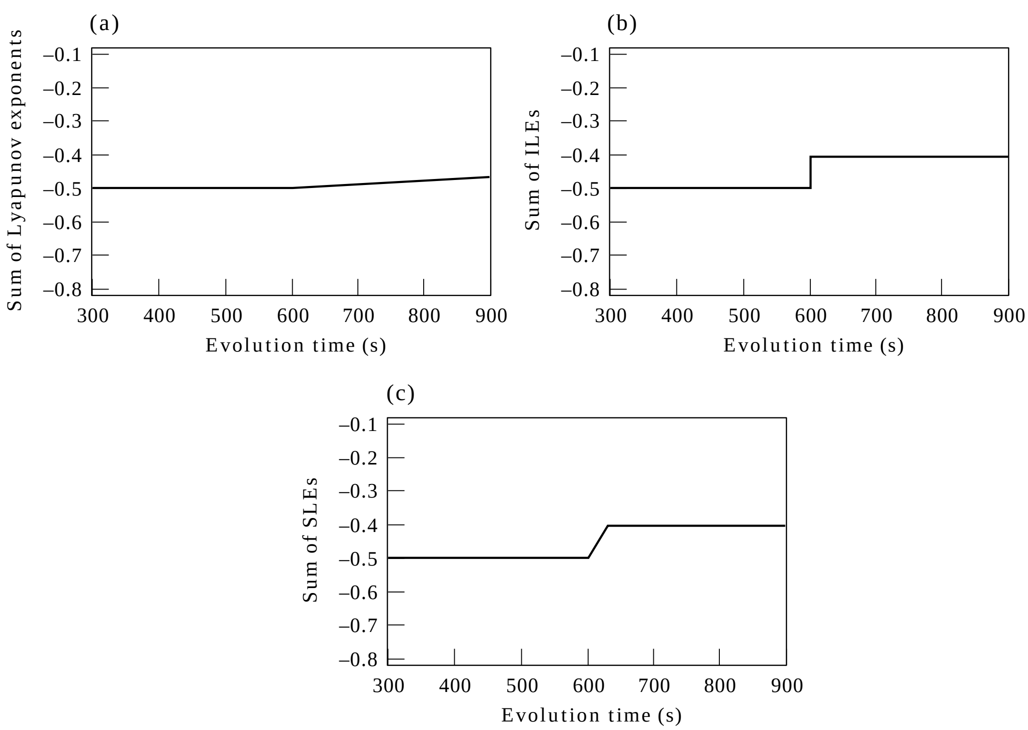

Figure 3. Results of the simulation for the Duffing equation ($A = 0{\cdot}4$, $k = 1$, $\omega = 1$, the damping parameter is changed from $c = 0{\cdot}5$ to $c = 0{\cdot}5$ at time 600 s. (a) The change of the sum of Lyapunov exponents is not easily distinguishable. (b) The change of the sum of ILEs is very distinctive. (c) The change of the sum of SLEs is clearly monitored: this is an averaged version of the sum of ILEs at every previous 10 forcing period.

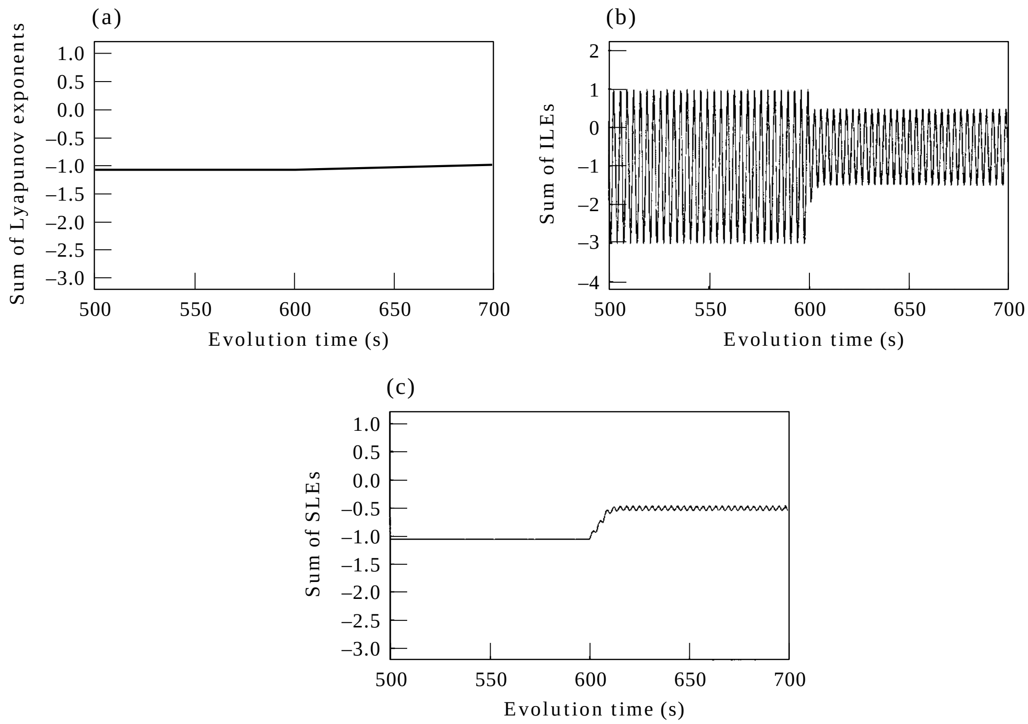

Figure 4. Results of the simulation for the Van der Pol equation ($k = 1$, the damping parameter is changed from $c = 1$ to $c = 0{\cdot}4$ at time 600 s). (a) The change of sum of Lyapunov exponents is not easily distinguishable. (b) The change of sum of ILEs are very distinctive. (c) The change of SLEs is clearly monitored: this is an averaged version of the sum of ILEs at every previous 10 forcing period.

EXAMPLE 2

As another example an autonomous system, the Van der Pol equation ($\ddot{x} + c(x^2 - 1)\dot{x} + kx = 0$) can be expressed in the form of (1)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y = f_1(x, t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -c(x^2 - 1)y - kx = f_2(x, y) \tag{23}$$

(N.B. this system is not chaotic). The generalised divergence of the flow is

$$\sum_{i=1}^{2} \frac{\partial f_1}{\partial x_1} = \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right] = -c(x^2 - 1). \tag{24}$$

This shows that the generalised divergence of the flow is continuously changing with time, i.e., it is the function of the variable x. Since the generalised divergence of the flow is equal to the sum of ILEs, the sum of ILEs is continuously varying with time and represents the continuously changing damping property of the system. On the other hand, the sum of Lyapunov exponents shows the average behaviour of the damping property of the system, and approaches the negative value of the damping coefficient. The sum of SLEs shows the short term averaged behaviour of the damping property of the system. The stiffness parameter is fixed and two different damping parameters are chosen in the simulation. Now, consider the system for which the damping parameter is 1 and is changed to 0·5 at certain times (at 600 s for this example). The results of this system are shown in Figure 4. In Figure 4(a), it can be seen that the sum of Lyapunov exponents varies very slowly when the damping parameter is changed, and so it is very difficult to see whether there are any changes at this time. On the other hand, in Figure 4(b), the sum of ILEs is clearly distinguishable at the right point when the damping parameter is changed. Also the sum of SLEs is presented in 4(c), and clearly demonstrates the ability to detect changes of damping. From the above two non-linear systems, it is shown that the ILEs or the SLEs has clear advantage over the Lyapunov exponents in monitoring the change of the damping property of a system.

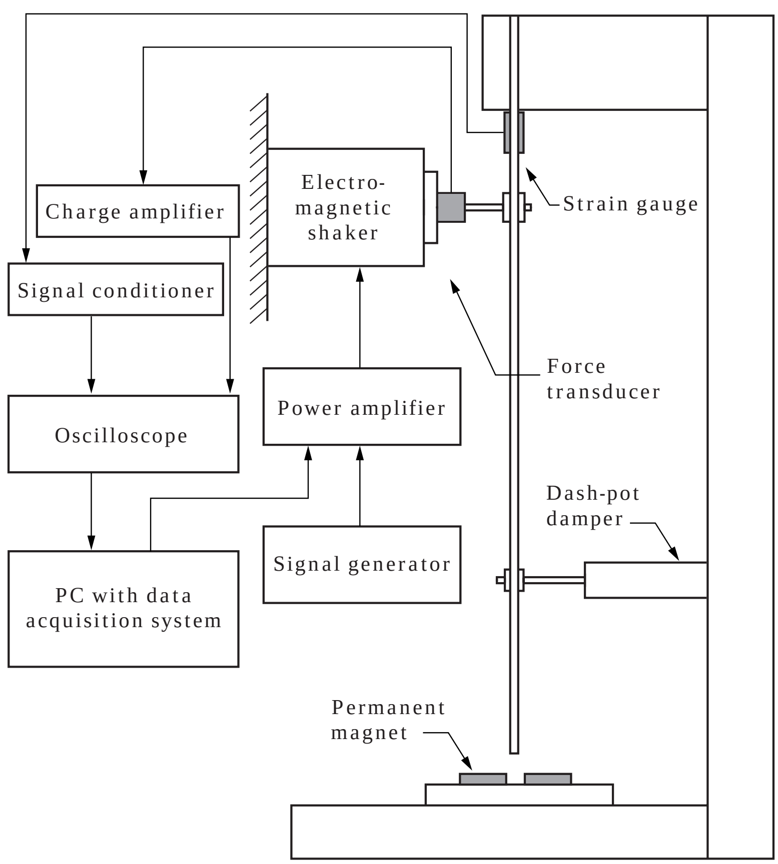

Figure 5. Experimental set-up.

5. EXPERIMENTAL RESULTS

The mechanical system studied here is a cantilever beam buckled by two magnets as illustrated in Figure 5. This system is similar to the "Moon Beam" in reference [11] in principle, and its dynamics are very similar to the Duffing type oscillator. However, unlike the "Moon Beam", the shaker is used directly on the cantilever beam and a dash-pot air damper is also introduced to control the amount of damping in the system. The damper is set to produce two cases by adjusting the damper plug giving high and low damping. The equations of motion for single degree of freedom approximation can be written as similar to equation (21), i.e.,

$$m\ddot{x} + c\dot{x} - \alpha x + \beta x^3 + A\cos(\omega t) \tag{25}$$

and details of equations of motion can be found in references [10, 11]. The generalised divergence of the flow is equal to the negative of normalised damping coefficient which is constant.

$$\sum_i \frac{\partial f_i}{\partial x_i} = -\frac{c}{m}\,. \tag{26}$$

The parameters are found by using the force-state mapping method [10]. The effective mass m is 0·77 kg, and the damping parameter c is 8·6 N · s/m for high damping and 3·0 N · s/m for low damping respectively.

This section demonstrates the use of the sum of ILEs for experimental data. From a measured time series (displacement signal x for this experiment), we estimate the ILEs, and then produce the SLEs to find whether there is a significant change of damping. As mentioned earlier, the sum of ILEs are noisy. Thus, the sum of SLEs are used by assuming that one can reduce noise by taking a time average. In order to obtain the Lyapunov exponents from a time series, it is necessary to reconstruct a phase portrait from a time series [12, 13]. From the reconstructed phase portrait, one can estimate the Lyapunov exponents. The various methods of the estimation of Lyapunov exponents can be found in [3, 4, 14–17], as well as ILEs in [9, 10]. A time series, is first measured and the phase portrait reconstructed by using the method of delays or by the SVD [12, 13]. From the reconstructed phase portrait, one can now estimate the SLEs.

Consider now the same two chaotic systems, i.e., one has high damping and the other has low damping. The reconstructed phase portraits by the SVD of both systems (high damping and low damping) are shown in Figures 6(a) and (b). Once the phase portraits are reconstructed, ILEs can be estimated as well as the SLEs. The sum of SLEs is shown in Figures 6(c) and (d) for the case of high damping and low damping respectively, and the average values of the sum of SLEs are $-10·1$ and $-4·5$ respectively. These average values are taken from several segments of the sum of SLEs, and the SLEs are calculated by averaging about 60 previous forcing periods of ILEs. From the sum of SLEs, the estimated damping parameters by using the effective mass (0·77 kg) are $c = 7·8$ N · s/m for the case of high damping and $c = 3·5$ N · s/m for the case of low damping. By considering the numerical errors on the estimation of the ILEs, the results show good

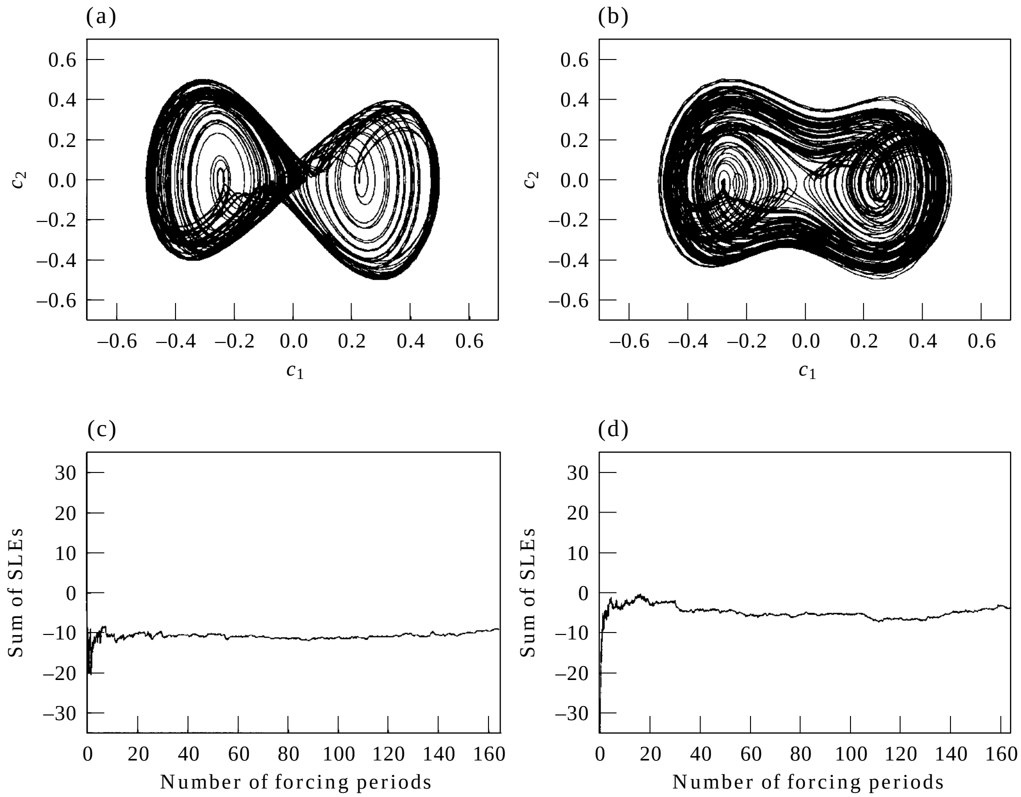

Figure 6. (a) Pseudo phase portrait by the SVD (high damping). (b) Pseudo phase portrait by the SVD (low damping). (c) Sum of SLEs (high damping). (d) Sum of SLEs (low damping).

agreement with the results of the force-state mapping method [10]. Now, consider that the damping of the system is changed at a certain time (at 130 forcing period in this case). The sum of Lyapunov exponents and the sum of SLEs are shown in Figure 7(a) and (b) respectively. As shown in Figure 7(a), the sum of Lyapunov exponents does not reveal the changes due to the nature of estimation of Lyapunov

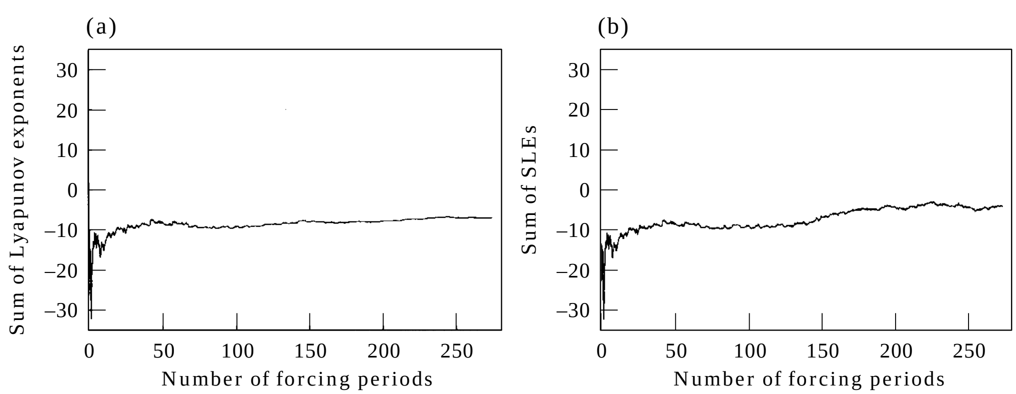

Figure 7. Damping of the system is changed at about 130 forcing periods. (a) Sum of Lyapunov exponents (the change is not clear). (b) Sum of SLEs (the change is clear).

exponents. On the other hand, one can see the distinctive change especially after about 150 forcing periods from the sum of SLEs as shown in Figure 7(b). The effect of size of the 'Window length (orbital periods)' for calculating the SLEs is not discussed in this paper. However, it is demonstrated in reference [10] that the smaller window length gives earlier detection while the larger window length gives less variation. A smaller window length with reliable estimation of SLEs would be ideal, however this is mainly depending on the algorithm used for the estimation. Algorithms for calculating the Lyapunov exponents from a time series are still a growing subject as are studies of the SLEs (and ILEs).

6. CONCLUDING REMARKS

Lyapunov exponents are very useful for the quantification of chaotic dynamics, but they represent only the average behaviour of the system, i.e., the Lyapunov exponent is a measure of exponential growth rate of nearby trajectories on average. On the other hand, the ILEs describe the instantaneous behaviour of the system, so when the characteristics of a system are subject to change it may be possible to monitor these, i.e., the ILE is a temporal measure and varies with time. As an example of the use of the ILEs and the sum of ILEs mechanical systems are used to detect any changes of the damping. Both the numerical results and the experimental results show the possibility of using ILEs in such variable conditions.

ACKNOWLEDGMENTS

The authors would like to acknowledge to the non-linear dynamics group of Professor F. W. Schneider at the University of Wuerzburg, Germany for making available of the software 'lce_exp' which computes Lyapunov exponents.

REFERENCES

1. F. C. Moon 1987 *Chaotic Vibrations*: *An Introduction for Applied Scientists and Engineers*. New York: Wiley.
2. A. Wolf, J. B. Swift, H. L. Swinney and J. A. Vastano 1985 *Physica D* **16**, 285–317. Determining Lyapunov exponents from a time series.
3. M. Sano and Y. Sawana 1985 *Physical Review Letters* **55**, 1082–1085. Measurement of the Lyapunov spectrum from a chaotic time series.
4. H. D. I. Abarbanel, R. Brown and M. B. Kennel 1991 *Journal of Nonlinear Science* **1**, 175–199. Vibration of Lyapunov exponents on a strange attractor.
5. G. L. Baker and J. P. Gollub 1990 *Chaotic Dynamics*: *An Introduction*. Cambridge University press.
6. P. Bergé, Y. Pomeau and C. Vidal 1984 *Order Within Chaos*: *Towards a Deterministic Approach to Turbulence*. Paris: Hermann.
7. S. D. Souza-Machado, R. W. Rollins, D. T. Jacobs and J. L. Hartmann 1990 *American Journal of Physics* **58**, 321–329. Studying chaotic systems using microcomputer simulations and Lyapunov exponent.
8. J. P. Eckman and D. Ruelle 1985 *Reviews of Modern Physics* **57**, 617–656. Ergodic theory of chaos and strange attractors.

9. K. SHIN and J. K. HAMMOND 1995 *2nd International Conference, Acoustical and Vibratory Surveillance* **1**, 125–136. Detection of changes in damping of nonlinear systems using Lyapunov exponents.
10. K. SHIN 1996 *Ph.D. thesis, University of Southampton*. Characterisation and identification of chaotic dynamical systems.
11. F. C. MOON and P. J. HOLMES 1979 *Journal of Sound and Vibration* **65**, 275–296. A magnetic strange attractor.
12. F. TAKENS 1980 *Lecture Notes in Mathematics* **898**, 365–381. Detecting strange attractors in turbulence.
13. D. S. BROOMHEAD and G. P. KING 1986 *Physica D* **20**, 217–236. Extracting qualitative dynamics from experimental data.
14. J. P. ECKMAN, D. RUELLE and S. CILIBERTO 1986 *Physical Review A* **34**, 4971–4979. Lyapunov exponents from time series.
15. T. M. KRUEL, M. EISWIRTH and F. W. SCHNEIDER 1993 *Physica D* **63**, 117–137. Computation of Lyapunov spectra: effect of interactive noise and application to a chemical oscillator.
16. K. BRIGGS 1990 *Physics Letters A* **151**, 27–32. An improved method for estimating Lyapunov exponents of chaotic time series.
17. R. BROWN, P. BRYANT and H. D. I. ABARBANEL 1991 *Physical Review A* **43**, 2787–2806. Computing the Lyapunov spectrum of a dynamical system from an observed time series.